$$-k(T_s)\left(\frac{\partial T}{\partial \upsilon}\right)_s = I \qquad (4)$$

where υ is the normal in transformed coordinates, ξ, η, ζ. This is a nonlinear problem, but the heating rate Q does not explicitly appear. The solution will be in the form $T = f(\xi, \eta, \zeta, \tau)$. Thus the temperature attained at the surface, with given material properties $k(T)$ and $c(T)$, will be the same for any heating rates Q whenever $\tau = Q^2 t$ is the same.

Drummond et al.[1] correlated their numerical results on the basis of the one-dimensional heat conduction equation of a semi-infinite solid with constant thermophysical properties

$$T_s - T_o = \frac{2t^{1/2} Q}{(k\rho c)^{1/2}_{\text{eff}} \pi^{1/2}} \qquad (5)$$

where T_o was the initial temperature. For a given temperature rise T, $\tau = Q^2 t$ is constant according to the analysis given above, so that $[t(T)]^{1/2} = [\tau(T)]^{1/2}/Q$

Thus

$$T_s - T_o = \frac{2[\tau(T)]^{1/2}}{(kc\rho)^{1/2}_{\text{eff}} \pi^{1/2}}$$

or

$$(k\rho c)^{1/2}_{\text{eff}} = \frac{2[\tau(T)]^{1/2}}{(T_s - T_o)\pi^{1/2}} \qquad (6)$$

The effect of temperature-dependent thermal conductivity $k(T)$ in the steady-state case may be dealt with through the Kirchoff transformation[2,3]

$$\theta = \int_{T_o}^{T} k(T_I)\, \mathrm{d}T_I \qquad (7)$$

For the transient case, this transformation leads to

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{\partial^2\theta}{\partial z^2} = \frac{\rho c(T)}{k(T)} \frac{\partial\theta}{\partial t} \qquad (8)$$

with the boundary condition

$$\left(\frac{\partial\theta}{\partial n}\right)_s = Q \qquad (9)$$

θ is a single-valued function of T depending only on the material properties through Eq. (7). Although this form is sometimes useful for numerical computations, it is of special interest analytically only for the steady-state case, in which it leads to Laplace's equation and in the transient case when $\rho c(T)/k(T)$ = constant [i.e., $c(T) \propto k(T)$], in which case it leads to the usual linear equation of transient heat conduction with θ in place of T. Note that the preceding transformation, which normalizes the equation and boundary condition with respect to Q, may also be applied in this case.

A completely analogous analysis to that which led to Eqs. (3-6) can be carried through for the case of heat transfer from a fluid at the surface with the boundary condition

$$-k(T_s)\left(\frac{\partial T}{\partial n}\right)_s = h(T_{aw} - T_s) \qquad (10)$$

where h is the heat-transfer coefficient and T_{aw} is the adiabatic wall temperature. In this case, Eq. (10) is divided through by h and the partial differential equation (1) is divided through by h^2. Analogously, the new time variable is $\tau = h^2 t$. Drummond et al. used the one-dimensional constant-property solution for this case, namely

$$[T_s(t) - T_s(o)] = [T_{aw} - T_s(o)][1 - e^{\gamma^2}\operatorname{erfc}\gamma] \qquad (11)$$

where

$$\gamma = \frac{ht^{1/2}}{(k\rho c)^{1/2}_{\text{eff}}}$$

Again, according to the foregoing theory, a given value of T_s is attained when $\tau = h^2 t$ is constant regardless of the value of h. Therefore, in this case also, Eq. (11) leads to a value of $(k\rho c)^{1/2}_{\text{eff}}$ which is a function of temperature only and not of heat transfer rate h.

If k, ρ, and c are functions of the coordinates x, y, and z as in the second set of tests reported in Ref. 1, the situation is not so simple. However, the transformations described above can still be useful. The thermophysical constants must also be transformed such that (e.g., in the case of constant heating rate Q)

$$k(x, y, z, T) = k(\xi/Q, \eta/Q, \zeta/Q, T) \qquad (12)$$

$$c(x, y, z, T) = c(\xi/Q, \eta/Q, \zeta/Q, T) \qquad (13)$$

Thus, the results for different values of Q give the same temperature rises at equal values of $Q^2 t = \tau$ only when the thermophysical properties are varied with Q in accordance with the transformation of the spatial coordinates. Although this situation is more complicated than when properties do not vary spatially, the results may be used to reduce the number of parameters involved in calculating numerically or testing systematically over ranges of heat-transfer rates and thermophysical property variations, if standard test model configurations are to be used.

References

[1] Drummond, J.P., Jones, R.A., and Ash, R.L., "Effective Thermal Property Improves Phase Change Paint Data," *AIAA Journal*, Vol. 14, Oct. 1976, pp. 1476-1478.

[2] Jakob, M., *Heat Transfer*, Vol. I., Wiley, New York, 1949, pp. 102-103.

[3] Dusenberre, G.M., *Heat Transfer Calculations by Finite Differences*, International Textbook, Scranton, Pa., 1961, pp. 172-173.

Reply by Authors to A. H. Flax

John Philip Drummond* and Robert A. Jones†
NASA Langley Research Center, Hampton, Va.
and
Robert L. Ash‡
Old Dominion University, Norfolk, Va.

THE authors wish to thank Dr. Flax for calling attention to a generalization of their work.[1] His discussion regarding the independence of the effective thermophysical property on the surface heat-transfer rate should reduce the effort needed when analyzing homogeneous wind-tunnel model materials. Several points should be noted, however.

First, we never intended that our analysis be limited to only thin-wing sections of models. In fact, the method is normally applied to "thick" model regions, and, as discussed in Ref. 2, more care must be taken when examining "thin" sections by

Received July 20, 1977.

Index category: Thermal Surface Properties.

*Aero-Space Technologist, Combustion Group, High-Speed Aerodynamics Division. Member AIAA.

†Head, Hypersonic Propulsion Branch, High-Speed Aerodynamics Division. Associate Fellow AIAA.

‡Associate Professor, Department of Mechanical Engineering and Mechanics.

Table 1 Dimensionless variables

	Reference time t_0	Reference thickness L_0
Temperature	$Tk_0/Q\sqrt{\alpha_0 t_0}$	Tk_0/QL_0
Thickness	$x_i/\sqrt{\alpha_0 t_0}$	x_i/L_0 $(i=1,2,3)$
Time	t/t_0	$\alpha_0 t/L_0^2$

limiting analysis time so as to not violate the semi-infinite slab assumption. Second, since thermal diffusivity is typically not constant for model materials, and Flax's transformed equations retain a nonlinear character for both homogeneous and inhomogeneous materials, a numerical analysis is still necessary to generate temperature-time histories for a calculation of the effective thermophysical property, $\beta=(k\rho c)^{1/2}_{\text{eff}}$. Flax's conclusion correctly indicates that only one program run at a single imposed heat flux is necessary to determine β at all heat fluxes for homogeneous materials. However, for inhomogeneous materials such as filled stycast plastic, it is doubtful that the thermophysical property variations could in general satisfy the requirement imposed by the spatial transformations.

Finally, some users may feel more comfortable with Flax's procedure if the variable transformations had been dimensionless. Because of the semi-infinite nature of the mathematical system, a somewhat artificial, although experimentally consistent, variable change could be made. Since those systems are either time limited (a maximum time duration over which the imposed heat flux is constant) or dimensionally limited (a minimum system thickness constrains the semi-infinite assumption), either a characteristic time t_0 or a characteristic thickness L_0 can be used. Then, employing the initial thermophysical properties of the front surface (thermal conductivity k_0, density ρ_0, specific heat c_0, and thermal diffusivity α_0) together with the temperature T, heat flux Q, and local thickness x_i, the dimensionally consistent transformations given in Table 1 can be used. No loss of generality occurs with those transformation sets and Flax's analysis still applies. The heat-transfer-coefficient analysis can be modified in a similar manner.

References

[1] Drummond, J. P., Jones, R. A., and Ash, R. L., "Effective Thermal Property Improves Phase Change Paint Data," *AIAA Journal*, Vol. 14, Oct. 1976, pp. 1476-1478.

[2] Drummond, J. P., "A Method for Improving the Accuracy in Phase Change Heat Transfer Data Through Increased Precision in Thermophysical Property Determination," M. E. Thesis, April 1975, Old Dominion University, Norfolk, Va.

Notice: SI Units

Starting in January 1978, manuscripts submitted to the *AIAA Journal* must use the International System of units, which is sometimes also called metric or Meter-Kilogram-Second system, for reporting *dimensional* results. At the author's discretion, he also may use English units in parentheses, following the SI units. This has no effect on papers or results which are reported in dimensionless form.

George W. Sutton
Editor-in-Chief